SOME CONVERGENT DEVELOPMENTS ASSOCIATED WITH IRREGULAR BOUNDARY CONDITIONS*

BY

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INTRODUCTORY NOTE. The author whose name appears at the head of this paper was a graduate student at Harvard when the United States entered the war. He went to an officers' training camp at the earliest opportunity, received a commission, and has been in France since the fall of 1917. Although his mathematical work was thereby interrupted before he was ready to regard even a first undertaking as definitely accomplished, some of the results that he had obtained are so striking, and, in their field, important, that I have asked and obtained his permission to publish them without further delay. Even of the work that he had done, only a part is contained in the manuscript that I now have, the rest having been lost in the confusion of his departure. The part that is preserved, forming the substance of the present paper, deals with a particular differential system, the place of which in a more general theory is indicated below. Mr. Hopkins had treated other particular systems in a similar way, and, when the work was suspended, was trying to raise the discussion a step higher in the scale of generality; he met with considerable difficulty in this attempt, and it is to be hoped that he will have an opportunity of coming back to the problem.

As the manuscript that I have is only a first rough draft, written before the declaration of war, with no thought of immediate publication, it would have required a considerable amount of editing, in any event. In view of this, I have taken the liberty of rewriting it altogether, in quite a different form,† which seems to me more elegant for the problem in hand, though less adapted to the sort of generalization that Mr. Hopkins had ultimately in view. The exposition, then, is the work of the undersigned, who must in particular assume responsibility for any imperfections that may be found in it; the substance, however, is essentially the work of Mr. Hopkins, and is rightfully published under his name.

DUNHAM JACKSON.

August 1, 1918.

1. PLACE OF THE PROBLEM IN A GENERAL THEORY

Birkhoff‡ has discussed in general terms the development of an arbitrary function $f(x)$ in a series of characteristic functions obtained from an ordinary linear differential equation of the nth order. He proves that if the boundary conditions associated with the differential equation are of the sort that he calls "regular," the expansion will converge, provided that $f(x)$ satisfies

* Presented to the Society, April 27, 1918.

† Mr. Hopkins adheres more closely to Birkhoff's method of treatment in the convergence proof, using the Green's function in determinant form throughout. The desirability of a more compact notation for the treatment of the present problem was urged by Professor Bôcher.

‡ These Transactions, vol. 9 (1908), pp. 373–395.

hypotheses of the order of those usually imposed for the sake of insuring convergence of a Fourier's series. It has been shown* that the distinction between regular and irregular boundary conditions is an essential one, and not merely an accident of Birkhoff's treatment. In a large class of problems with irregular conditions, the expansion of a function having merely a limited number of continuous derivatives can not possibly converge in the usual way,† and the expansion even of an analytic function is likely to be rapidly divergent. In order to lead to a uniformly convergent series, the function $f(x)$ must be of decidedly more special type. The following discussion reveals the exact nature of the specialization, in the case of the series given by a particular differential equation of the third order, with a particular set of irregular boundary conditions. While the scope of the paper is therefore decidedly limited, in comparison with the extent of the field that opens itself for investigation, it gives for the first time a constructive account of a type of series which have some of the properties of power series on the one hand and of trigonometric series on the other, and which possess further novel and striking properties of their own.

2. Necessary condition for convergence

The starting point of the discussion is the differential equation‡

$$\frac{d^3u}{dx^3} + \rho^3 u = 0,$$

with the boundary conditions

$$u(0) = 0, \qquad u'(0) = 0, \qquad u(\pi) = 0.$$

Its characteristic numbers are real, and are distributed in accordance with the formula

$$\rho_n = C + \frac{2n}{\sqrt{3}} + \epsilon_n,$$

where C is independent of n, and $\lim_{n=\infty} \epsilon_n = 0$. With the notation

$$\delta(t) = e^{-t} + \omega e^{-\omega t} + \omega^2 e^{-\omega^2 t}, \tag{1}$$

where ω is one of the complex cube roots of unity,

$$\omega = e^{2\pi i/3},$$

* D. Jackson, Proceedings of the American Academy, vol. 51 (1915–16), pp. 383–417.

† That is, can not converge uniformly to the value of the function.

‡ For the introductory statements about this differential system, cf. Jackson, loc. cit., pp. 384–393.

the nth characteristic function is

$$u_n(x) = \delta(\rho_n x). \tag{2}$$

The problem is to determine under what circumstances the formal development of an arbitrary function $f(x)$ in a series of the form

$$\sum_{n=1}^{\infty} a_n u_n(x) \tag{3}$$

will converge uniformly to the value of the function. We shall obtain a necessary condition on the function $f(x)$, and then shall show that, in their principal features at least, hypothesis and conclusion can be interchanged. It is admitted that a clean-cut necessary and sufficient condition is beyond the limits of this paper; but an essential part of such a condition, to all appearances the most significant part, is clearly brought to light.

In establishing the necessity of the condition, the differential system which led to the series, and the corresponding theory, for which reference has been made to another paper, may be left altogether in the background. For following all the work that is done here, it is enough to take the series (3) on its own merits, with the explicit formula for $u_n(x)$, and the observation that ρ_n is a positive real number, of the order of magnitude of n.

Suppose that the series (3) converges uniformly throughout an interval $x_1 \leqq x \leqq x_0$, however small, where $0 \leqq x_1 < x_0$. We shall be concerned primarily with the behavior of the series in the interval $(0, \pi)$, but for the purposes of the present section, x_0 may be less than, equal to, or greater than π. There will be a constant g such that

$$|a_n u_n(x)| \leqq g \tag{4}$$

throughout the interval, for all values of n. Although the expression for $u_n(x)$ given above involves imaginaries, the function itself is real, and may be represented by the equivalent formula

$$u_n(x) = e^{-\rho_n x} + 2e^{\frac{1}{2}\rho_n x} \cos\left(\frac{\sqrt{3}}{2}\rho_n x - \frac{2\pi}{3}\right).$$

The first term on the right is exceedingly small, for large values of n; the order of magnitude of the whole expression is that of $e^{\rho_n x/2}$. More precisely, $u_n(x)$ has this order of magnitude, unless the trigonometric factor happens to be zero or nearly zero. But if n is large, the oscillations of the cosine will be so rapid that there will surely be at least one point x_n' between x_1 and x_0 at which it is equal to $+1$, and at such a point (even the small term being positive)

$$u_n(x_n') > 2e^{\frac{1}{2}\rho_n x_n'} > 2e^{\frac{1}{2}\rho_n x_1}.$$

Since (4) holds for $x = x_n'$, it may be inferred, for each value of n from a certain point on, that

(5) $$|a_n| < \tfrac{1}{2}ge^{-\frac{1}{2}\rho_n x_1}.$$

The number x_1 was supposed given. But if the series converges uniformly for $x_1 \leqq x \leqq x_0$, it converges uniformly for $\xi_1 \leqq x \leqq x_0$, if ξ_1 is any number between x_1 and x_0; and the reasoning may be applied to the interval (ξ_1, x_0). So the inequality (5) holds, if x_1 is replaced by ξ_1, except that more terms may have to be left out of account at the beginning. Even these terms, finite in number, may be brought under the formula, if a suitably chosen larger constant, presumably dependent on ξ_1, is substituted for $g/2$. With a slight change in notation—since we have been saying, practically, that x_1 might have had any larger value, below x_0, in the first place, and the inequality is merely strengthened if x_1 is replaced by a smaller value—we may assert that

If the series (3) *converges uniformly throughout any interval reaching out to the point* x_0, *and if* x_1 *is any number less than* x_0, *there will be a number* h, *independent of* n, *such that*

$$|a_n| < he^{-\frac{1}{2}\rho_n x_1}$$

for all values of n.

It will be seen that such a restriction on the coefficients implies convergence of the series for an extended range of complex as well as of real values of x.

Let x_2 be any positive number less than x_1. Setting $x = \xi + \eta i$, consider the magnitude of the three terms which make up $u_n(x)$, according to (1) and (2). We shall have

(6) $$|e^{-\rho_n x}| = e^{-\rho_n \xi} \leqq e^{\frac{1}{2}\rho_n x_2}$$

if $\xi \geqq -\frac{1}{2}x_2$, that is, if x is any point on or to the right of the vertical line $\xi = -\frac{1}{2}x_2$. Similarly,

(7) $$|e^{-\omega\rho_n x}| \leqq e^{\frac{1}{2}\rho_n x_2},$$

provided that ωx satisfies the same condition, that is, provided that x is on or below the line obtained by rotating the vertical line about the origin as a center through 120° in the negative direction. In the same way,

(8) $$|e^{-\omega^2\rho_n x}| \leqq e^{\frac{1}{2}\rho_n x_2},$$

if x is on or above the line into which the vertical line is carried by a rotation of 120° in the positive direction.* These three lines bound an equilateral triangle, with its center at the origin, and with one vertex at the point x_2. If x is within or on the boundary of the triangle, the relations (6), (7), and (8)

* More formally,

$$|e^{-\rho_n x}| = e^{-\rho_n \xi}, \qquad |e^{-\omega\rho_n x}| = e^{\frac{1}{2}\rho_n(\xi+\sqrt{3}\eta)}, \qquad \text{and} \qquad |e^{-\omega^2\rho_n x}| = |e^{\frac{1}{2}\rho_n(\xi-\sqrt{3}\eta)}|;$$

and these three quantities will be less than or equal to $e^{\rho_n x_2/2}$, if $\xi \geqq -\frac{1}{2}x_2$, $\xi + \sqrt{3}\eta \leqq x_2$, $\xi - \sqrt{3}\eta \leqq x_2$, respectively.

all subsist at once;

$$|u_n(x)| \leqq 3e^{\frac{1}{2}\rho_n x_2},$$

and

$$|a_n u_n(x)| \leqq 3he^{\frac{1}{2}\rho_n(x_2 - x_1)}.$$

Since $x_2 - x_1$ is negative, and since ρ_n is of the order of magnitude of n, the right-hand member of the last inequality is the general term of a convergent series of positive terms. This means that the series (3) converges uniformly throughout the triangle. It will be recalled that x_2 is any number less than x_1, and x_1 any number less than x_0. Therefore we may say that

The series represents a function which is analytic throughout the interior of an equilateral triangle having its center at the origin, and one vertex at the point x_0.

It is to be noticed that if X is the greatest possible value of x_0, or the upper limit of possible values of x_0, the corresponding triangle is the actual region of convergence of the series, except possibly for points on the positive axis of reals, and on the rays inclined at 120° and 240°, beyond the range of uniform convergence. That is, the series can not converge at any point outside the triangle and not lying on one of these three rays. Suppose it were convergent at such a point, x_2'. For $x = x_2'$, one of the three terms of $u_n(x)$ will be so much larger than either of the others as to determine the order of magnitude of the whole expression; there is no possibility of two terms destroying each other. Let the triangle determined by X be magnified until one side passes through x', and let X' be the right-hand vertex of the new triangle. The term of $u_n(x)$ which is largest at x_2' will have the same absolute value all along the side through x_2', and along each of the other two sides the term which outweighs the others will have this same absolute value still. Since the absolute value of the preponderant term is essentially the absolute value of $u_n(x)$, this means that $u_n(x)$ has the same order of magnitude all around the perimeter of the triangle, except that it may be *smaller* at a vertex, because of the possibility of cancellation there. In particular, $u_n(X')$ does not exceed $u_n(x_2')$ in order of magnitude. Since $a_n u_n(x_2')$ remains finite, the same is true of $a_n u_n(X')$. Therefore, by reasoning similar to that already employed, the series converges uniformly for $0 \leqq x \leqq x_0'$, if x_0' is any number less than X'; and this is inconsistent with the hypotheses, since x_0' might be taken greater than X.

Let the sum of the series be denoted by $f(x)$. Further properties of this function, beyond its mere analytic character, can be deduced from the special form of the individual terms. If any one of the functions $u_n(x)$ is expanded in a power series, by combining the series for the three exponentials of which it is made up, two thirds of the terms will cancel, because $1 + \omega + \omega^2 = 0$, and only the second, fifth, eighth, . . ., powers of x will remain. So it appears, formally at least, that $f(x)$ will be similarly constituted. This may be verified as follows.

In the first place, $f(0) = f'(0) = 0$, because $u_n(0) = u_n'(0) = 0$, for every value of n, and the series can be differentiated term by term. It follows that we can write

$$f(x) = x^2 \phi(x),$$

where $\phi(x)$ is analytic in the same region as $f(x)$. By direct substitution it it seen that $u_n(\omega x) = \omega^2 u_n(x)$, for all values of n, whence $f(\omega x) = \omega^2 f(x)$. For ϕ this means that

$$\phi(\omega x) = f(\omega x)/(\omega^2 x^2) = f(x)/x^2 = \phi(x),$$

and so it may be inferred* that $\phi(x)$ can be represented by a power series in x^3.

Thus the necessary condition with regard to the form of $f(x)$ is established. We may express it by saying that

$$f(x) = x^2 \psi(x^3),$$

where ψ is a single-valued analytic function of its argument, or by saying that $f(x)$ is analytic, $f(0) = f'(0) = 0$, and $f''(x)$ is a single-valued function of x^3. The necessity of the condition, in an appropriate region, has been proved for uniform convergence over any interval of the positive axis of reals. To separate the results most closely related to the remainder of the discussion from those to which we shall not have occasion to refer again, we may summarize in two stages, thus

THEOREM I. *If the series* (3) *converges uniformly throughout the interval* $0 \leqq x \leqq x_0$, *where* $0 < x_0 \leqq \pi$, *it represents a function* $f(x)$ *which is analytic throughout the interior of an equilateral triangle having its center at the origin and one vertex at the point* $x = x_0$, *and which involves in its power series expansion only powers of index congruent to* 2, *mod.* 3.

THEOREM I*a*. *In the hypothesis of Theorem I, the interval* $(0, x_0)$ *may be replaced by any interval* (x_1, x_0), *where* $0 < x_1 < x_0$, *and the restriction* $x_0 \leqq \pi$ *is unnecessary.*

3. PROOF OF CONVERGENCE FOR THE FUNCTION x^2

The simplest function satisfying the necessary conditions of Theorem I is the function x^2. The purpose of this section is to prove

THEOREM II. *The formal development of* x^2 *in a series of the form* (3) *converges uniformly to the value* x^2 *for*† $0 \leqq x \leqq x_0$, *if* $0 < x_0 < \pi$.

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* Let $\varphi(x)$ be expanded according to powers of x; since $\varphi(\omega^2 x) = \varphi(\omega x) = \varphi(x)$, we may write $\varphi(x) = \frac{1}{3}[\varphi(x) + \varphi(\omega x) + \varphi(\omega^2 x)]$, and if we substitute the series for φ inside the bracket, all the powers of x except the third, sixth, ninth, $\cdots$, will cancel.

† The series of course converges to the value 0 for $x = \pi$, since each term vanishes separately; and from this it follows that the convergence can not be uniform for the whole interval $(0, \pi)$.

We must refer to the general theory of boundary value and expansion problems once more for the formulas on which the proof is based. But here again, as in the preceding section, we can write down the necessary formulas at the start, and then use them on their own merits, without further reference to their derivation.

The coefficients in the formal expansion of an arbitrary function $f(x)$ in a series (3) are given by integrals analogous to those for the coefficients in a Fourier's series; it is not necessary even to write them down here. On the other hand, the following expression for the sum of the first n terms of the series is essential*:

$$I_n(x) = \frac{1}{2\pi i}\int_{\gamma_n} d\rho \int_0^{\pi} 3\rho^2 G(x, s, \rho) f(s)\, ds. \tag{9}$$

Here γ_n indicates a path in the ρ-plane, consisting of an arc of 120° of a circle about the origin, of such radius as to include just the first n of the characteristic numbers ρ_i. It may be thought of as extending in the positive direction from $-R_n\omega$ to $-R_n\omega^2$, say, if R_n is the radius of the circle. It corresponds to a complete circuit in the plane of the variable ρ^3, which is the parameter that actually appears in the differential equation. The function G is the Green's function of the differential system. Its form may be found by evaluating the general determinant expression † for such a function, as applied to the problem in hand. The reduction to a compact formula involves a certain amount of computation, but it is fairly straightforward, and there is perhaps no need of giving the details of the process.‡ The result is conveniently expressed in terms of the function δ, introduced in (1). It has different forms, according as x is greater or less than s:

$$(10)\qquad \begin{aligned} 3\rho^2 G &= \frac{\delta(\overline{\rho x - s})\,\delta(\rho\pi) - \delta(\overline{\rho\pi - s})\,\delta(\rho x)}{\delta(\rho\pi)} \qquad &\text{if} \quad x > s, \\ 3\rho^2 G &= -\frac{\delta(\overline{\rho\pi - s})\,\delta(\rho x)}{\delta(\rho\pi)} &\text{if} \quad x < s. \end{aligned}$$

* Birkhoff, loc. cit., pp. 379, 390.

† Birkhoff, loc. cit., p. 378.

‡ Some of the principal steps are these: Using the notation and formulas of Birkhoff, loc. cit., pp. 378, 391, let $y_1(x) = e^{-\rho x}$, $y_2(x) = e^{-\omega\rho x}$, $y_3(x) = e^{-\omega^2\rho x}$. It is found that $\bar{y}_1(s) = e^{\rho s}/(3\rho^2)$, $\bar{y}_2(s) = \omega e^{\omega\rho s}/(3\rho^2)$, $\bar{y}_3(s) = \omega^2 e^{\omega^2\rho s}/(3\rho^2)$, so that

$$\bar{G} = \pm\,\delta(\overline{\rho x - s})/(6\rho^2),$$

the sign being $+$ for $x > s$, $-$ for $x < s$. The values of the functions $\bar{y}_i$ are most easily verified by noticing that they satisfy the equations $\Sigma y_i\bar{y}_i = 0$, $\Sigma y_i'\bar{y}_i = 0$, $\Sigma y_i''\bar{y}_i = 1$, the index i running from 1 to 3. The value of $\bar{G}$ is substituted in the formula for G. In the determinant which forms the numerator, suitable multiples of the first three columns are added to the last column, so as to reduce the second and third of its elements to zero, and, if $x < s$, the first element as well. It is easy then to evaluate the determinant and finish the computation.

It will be seen that these expressions differ only by an additive term $\delta(\overline{\rho x - s})$; but it is convenient to write them, as we have done, with a common denominator throughout.

The problem is, *with the formulas* (10) *for the Green's function, and with the substitution of* s^2 *for* $f(s)$ *in* (9), *to show that*

$$\lim_{n=\infty} I_n(x) = x^2,$$

uniformly in $(0, x_0)$.

The integration with regard to s in (9) can be performed explicitly. In (10), only the quantities $\delta(\overline{\rho x - s})$ and $\delta(\overline{\rho \pi - s})$ depend on s; the latter appears in the integrand throughout the whole interval from 0 to π, the former only while s goes from 0 to x. By integration by parts,

$$\int_0^x s^2 e^{\rho(s-x)}\, ds = \frac{x^2}{\rho} - \frac{2x}{\rho^2} + \frac{2}{\rho^3} - \frac{2}{\rho^3} e^{-\rho x},$$

$$\int_0^x s^2 e^{\omega\rho(s-x)}\, ds = \frac{x^2}{\omega\rho} - \frac{2x}{\omega^2\rho^2} + \frac{2}{\rho^3} - \frac{2}{\rho^3} e^{-\omega\rho x},$$

$$\int_0^x s^2 e^{\omega^2\rho(s-x)}\, ds = \frac{x^2}{\omega^2\rho} - \frac{2x}{\omega\rho^2} + \frac{2}{\rho^3} - \frac{2}{\rho^3} e^{-\omega^2\rho x},$$

so that, with the relation $1 + \omega + \omega^2 = 0$ again,

$$\int_0^x s^2 \delta(\overline{\rho x - s})\, ds = \frac{3x^2}{\rho} - \frac{2}{\rho^3}\delta(\rho x).$$

As far as this computation is concerned, x might have any value whatever; in particular,

$$\int_0^\pi s^2 \delta(\overline{\rho \pi - s})\, ds = \frac{3\pi^2}{\rho} - \frac{2}{\rho^3}\delta(\rho \pi).$$

It is at this point that the very special form of the function to be developed has its effect. It we had performed the same calculation with 1 or s in place of s^2, we should not have found precisely $\delta(\rho x)$ and $\delta(\rho\pi)$ for the concluding terms on the right, but other combinations of the exponentials instead, and the cancellation of these terms at the next stage, essential for the success of the convergence proof, would not take place. As it is, we see that

$$\begin{aligned}\delta(\rho\pi)\int_0^x s^2 \delta(\overline{\rho x - s})\, ds - \delta(\rho x)\int_0^\pi s^2 \delta(\overline{\rho\pi - s})\, ds \\ = \frac{3x^2}{\rho}\delta(\rho\pi) - \frac{3\pi^2}{\rho}\delta(\rho x),\end{aligned} \tag{11}$$

that is,

$$\int_0^\pi 3\rho^2 G s^2\, ds = \frac{3x^2}{\rho} - \frac{3\pi^2}{\rho}\cdot\frac{\delta(\rho x)}{\delta(\rho\pi)}, \tag{12}$$

and the most formidable terms have disappeared. It is to be remembered that the computation so far is exact; we have not dismissed as negligible anything that is actually present. We are ready now, however, to examine carefully the order of magnitude of the quantities that we are dealing with.

Let $\delta(\rho\pi)$ be written in the form

$$\delta(\rho\pi) = e^{-\rho\pi} + \omega e^{-\omega\rho\pi}(1 + \omega e^{\sqrt{3}\rho\pi i}).$$

The parenthesis is a periodic function of ρ, having the real period $2/\sqrt{3}$. It becomes infinite uniformly as the imaginary part of ρ becomes negatively infinite, and uniformly approaches 1 when the imaginary part of ρ becomes positively infinite. It vanishes at the points

$$\rho_n' = \frac{1}{3\sqrt{3}} + \frac{2n}{\sqrt{3}}, \qquad n = 0, \pm 1, \pm 2, \cdots,$$

and nowhere else in the ρ-plane. Consequently, if small circles of constant radius ϵ are described about the points ρ_n', and the interiors of these circles are removed from the ρ-plane, the absolute value of the parenthesis will have a positive minimum k_1 in the part of the plane that is left.

Let S' denote the sector of the ρ-plane for which $-\pi/3 \leqq \arg\rho \leqq 0$, and S'' the sector for which $0 \leqq \arg\rho \leqq \pi/3$. Each of the arcs γ_n crosses both of these sectors; let its halves be denoted by γ_n' and γ_n'' respectively. For large values of n, the characteristic numbers ρ_n will be inside the little circles of radius ϵ mentioned in the preceding paragraph,* and the arcs may be supposed drawn so as not to cut any of these circles. On γ_n'', $e^{-\rho\pi}$ approaches zero uniformly, and $e^{-\omega\rho\pi}$ becomes uniformly infinite, as n increases without limit. So, if k is any positive constant less than k_1,

$$|\delta(\rho\pi)| > k|e^{-\omega\rho\pi}|$$

on this arc, for all values of n from a certain point on.

On the other hand,

$$|\delta(\rho x)| \leqq 3|e^{-\omega\rho x}| \leqq 3|e^{-\omega\rho x_0}|$$

for $0 \leqq x \leqq x_0$ and for ρ in S''. On γ_n'', therefore,

$$\left|\frac{\delta(\rho x)}{\delta(\rho\pi)}\right| < \frac{3}{k}\,|e^{\omega\rho(\pi - x_0)}|,$$

which approaches zero uniformly along the arc. Consequently

$$\lim_{n=\infty}\int_{\gamma_n''}\frac{\delta(\rho x)}{\rho\delta(\rho\pi)}\,d\rho = 0,$$

* It is not asserted that ρ_n will be in the circle about the point ρ_n' with the same subscript n. So far as the writer knows, there may be a constant difference between these subscripts, but that is inessential for the main argument.

uniformly for $0 \leqq x \leqq x_0$; the factor ρ in the denominator would compensate for the increase in the length of the arc with n, even if the convergence of the exponential were not already strong enough without it.

The conclusion can be extended at once to the integral over γ_n', since the integrand takes on conjugate imaginary values for conjugate imaginary values of ρ.

The proof of the theorem is now practically immediate. For the ρ-integral of the first term on the right in (12) is independent of n:

$$\int_{\gamma_n} \frac{3x^2}{\rho} d\rho = 3x^2 \int_{\gamma_n} \frac{d\rho}{\rho} = 3x^2 [\log \rho]_{-R_n\omega}^{-R_n\omega^2} = 3x^2 \cdot \frac{2\pi i}{3};$$

so that, uniformly for $0 \leqq x \leqq x_0$,

$$\lim_{n=\infty} \int_{\gamma_n} d\rho \int_0^\pi 3\rho^2 Gs^2 \, ds = 2\pi i x^2,$$

which is equivalent to the relation to be proved.

It is scarcely more difficult to prove the convergence for x^5 or x^8, or any other single power of the form x^{3m+2}. For example,

$$\int_0^x s^5 \, \delta(\overline{\rho x - s}) \, ds = \frac{3x^5}{\rho} - \frac{3 \cdot 5 \cdot 4 \cdot 3x^2}{\rho^4} + \frac{5!}{\rho^6} \delta(\rho x),$$

$$\int_0^\pi 3\rho^2 Gs^5 \, ds = \frac{3x^5}{\rho} - \frac{3\pi^5}{\rho} \frac{\delta(\rho x)}{\delta(\rho\pi)} - \frac{3\cdot5\cdot4\cdot3x^2}{\rho^4} + \frac{3\cdot5\cdot4\cdot3\pi^2}{\rho^4} \frac{\delta(\rho x)}{\delta(\rho\pi)}. \tag{13}$$

When integrated with regard to ρ, the first term on the right in (13) gives $2\pi i x^5$. The second and fourth terms give results that approach zero, because of the relative magnitudes of $\delta(\rho x)$ and $\delta(\rho\pi)$. In the case of the third term, the magnitude of the denominator would be enough to bring about convergence, even without the fact that

$$\int_{\gamma_n} \frac{d\rho}{\rho^4} = 0$$

exactly. So there is no new difficulty here, except the greater length of the formulas.

The methods of the present section, then, are sufficient to establish the convergence of the formal development for any polynomial involving only the powers of x designated in Theorem I. For a function represented by an infinite series of these powers, some new considerations are needed, which it is the purpose of the following section to supply.

4. PROOF OF CONVERGENCE FOR THE GENERAL FUNCTION $x^2 \psi(x^3)$

The final theorem of the paper is the following.

THEOREM III. *If $f(x)$ is any function which is analytic throughout the interior and on the boundary of a circle of radius x_0 about the origin, where $0 < x_0 < \pi$, and which involves in its power series expansion only powers of index congruent to 2, mod. 3, and which has furthermore a continuous second derivative for real values of x throughout the whole interval $0 \leqq x \leqq \pi$, then the formal development of $f(x)$ converges uniformly to the value of the function for $0 \leqq x \leqq x_0$.*

In various respects this falls short of being a complete converse of Theorem I. It restricts x_0 to values less than π, necessarily, as was seen in connection with the function x^2; but more than this, it presupposes the analytic character of $f(x)$, not merely throughout the triangle of Theorem I, but throughout the circle circumscribed about this triangle; and it includes an assumption relative to the interval (x_0, π). It would not be hard to do something toward reducing these discrepancies, but in the present paper we shall be satisfied with the theorem as stated.* Clearly some assumption must be made covering the whole interval $(0, \pi)$, in order that the integrals involved may have a meaning.

We use the general formulas of the preceding section, in particular the expression (9) for the sum of n terms of the series, and follow the same method of procedure as far as may be.

It is no longer possible to carry through the s-integration to a finish. Integrating by parts twice, and remembering that $f(0) = f'(0) = 0$, we find that

$$\int_0^x f(s)\, e^{\rho(s-x)}\, ds = \frac{1}{\rho} f(x) - \frac{1}{\rho^2} f'(x) + \frac{1}{\rho^2}\int_0^x f''(s)\, e^{\rho(s-x)}\, ds,$$

with corresponding formulas for the integrals involving $e^{\omega\rho(s-x)}$ and $e^{\omega^2\rho(s-x)}$. By combination of the three equations,

$$\int_0^x f(s)\, \delta(\overline{\rho x - s})\, ds = \frac{3f(x)}{\rho} + \frac{1}{\rho^2}\int_0^x f''(s)\,[\, e^{\rho(s-x)} + \omega^2 e^{\omega\rho(s-x)} + \omega e^{\omega^2\rho(s-x)}\,]\, ds.$$

The bracket on the right differs from a δ-function by the interchange of the coefficients ω and ω^2 before the exponentials. Let it be denoted by $\delta_1(\overline{\rho x - s})$. Since the computation is valid† for $x = \pi$,

* With regard to these secondary hypotheses, Mr. Hopkins's own statement is in some respects less general and in some respects more general than that in the text. The writer must admit that he has not studied the proof by the other method carefully enough to be sure how far the differences are significant for the respective methods, and how far they are accidental.

† It is for the integration by parts at this stage that the second derivative of f is used all the way from 0 to π.

$$\delta(\rho\pi)\int_0^x f(s)\,\delta(\overline{\rho x - s})\,ds - \delta(\rho x)\int_0^\pi f(s)\,\delta(\overline{\rho\pi - s})\,ds$$

$$(14)\qquad = \frac{3f(x)}{\rho}\,\delta(\rho\pi) - \frac{3f(\pi)}{\rho}\,\delta(\rho x) + \frac{1}{\rho^2}\left[\delta(\rho\pi)\int_0^x f''(s)\,\delta_1(\overline{\rho x - s})\,ds - \delta(\rho x)\int_0^\pi f''(s)\,\delta_1(\overline{\rho\pi - s})\,ds\right].$$

This corresponds to the formula (11) in the preceding section, except that for $f(s) = s^2$ the bracket reduces to zero. It will be found possible in the present case, by making use of the special properties of the function f, to break up the bracket into a number of terms, so that two of them will cancel, while the others are of such order of magnitude as not to interfere with the convergence.

Before undertaking this, however, let us dispose of the simpler terms in (14). Taking account of the denominator in (10), and denoting the bracket in (14) by A, we have

$$\int_0^\pi 3\rho^2\, Gf(s)\,ds = \frac{3f(x)}{\rho} - \frac{3f(\pi)}{\rho}\cdot\frac{\delta(\rho x)}{\delta(\rho\pi)} + \frac{A}{\rho^2\,\delta(\rho\pi)}.$$

All this has to be integrated with regard to ρ over the arc γ_n, to give $2\pi\, I_n(x)$. The result of integrating $3f(x)/\rho$ is $2\pi i f(x)$ exactly. The ρ-integral of $3f(\pi)\,\delta(\rho x)/[\rho\delta(\rho\pi)]$ approaches zero, uniformly for $0 \leqq x \leqq x_0$, when n becomes infinite, by the reasoning of the preceding section. It remains to show that the integral of $A/[\rho^2\,\delta(\rho\pi)]$ also converges to zero.

Let the first integral in the bracket A be written as a sum of three, involving respectively the three terms of $\delta_1(\overline{\rho x - s})$. Consider the second of these integrals

$$\int_0^x f''(s)\cdot\omega^2 e^{\omega\rho(s-x)}\,ds = \omega e^{-\omega\rho x}\int_0^x f''(s)\,e^{\omega\rho s}\cdot\omega\,ds.$$

We begin now to make use of the properties of $f(s)$ in the complex s-plane. If a new variable is introduced by the transformation $s' = \omega s$, changing the integral along the axis of reals into an integral along a ray inclined at 120°, *$f''(s)$ is invariant under the transformation, being a single-valued function of s^3.* Dropping the accent after performing the transformation, we may write

$$\omega e^{-\omega\rho x}\int_0^x f''(s)\,e^{\omega\rho s}\cdot\omega\,ds = \omega e^{-\omega\rho x}\int_0^{\omega x} f''(s)\,e^{\rho s}\,ds.$$

Similarly,

$$\int_0^x f''(s)\cdot\omega e^{\omega^2\rho(s-x)}\,ds = \omega^2\, e^{-\omega^2\rho x}\int_0^{\omega^2 x} f''(s)\,e^{\rho s}\,ds;$$

while, without any change of variable,

$$\int_0^x f''(s)\,e^{\rho(s-x)}\,ds = e^{-\rho x}\int_0^x f''(s)\,e^{\rho s}\,ds.$$

All three integrals are now expressed with a common integrand.

A further transformation is rendered possible by the fact that $f(s)$, *and so* $f''(s)$, *is an analytic function of* s. For any particular values of $x\,(0 \leqq x \leqq x_0)$ and ρ, let y be the point defined by the conditions

$$|y| = |x|, \qquad \arg y = \pi - \arg \rho,$$

so that

$$\rho y = -\,|\rho x|.$$

The straight path from 0 to ωx, for example, may be replaced by a broken line, extending from 0 to y and then from y to ωx. The two paths and the triangle bounded by them will be wholly interior to the region where the analytic character of $f(s)$ is assumed, so that the change of path will not change the value of the integral. That is,

$$\int_0^x f''(s)\cdot\omega^2 e^{\omega\rho(s-x)}\,ds = \omega e^{-\omega\rho x}\int_0^y f''(s)\,e^{\rho s}\,ds + \omega e^{-\omega\rho x}\int_y^{\omega x} f''(s)\,e^{\rho s}\,ds.$$

The paths from 0 to $\omega^2 x$ and from 0 to x may be modified in the same way, the first segment of the new path extending from 0 to y in each case. The result of all this is that

$$\begin{aligned}(15)\qquad &\int_0^x f''(s)\,\delta_1(\overline{\rho x - s})\,ds\\ &= \left[\delta(\rho x)\int_0^y + e^{-\rho x}\int_y^x + \omega e^{-\omega\rho x}\int_y^{\omega x} + \omega^2 e^{-\omega^2\rho x}\int_y^{\omega^2 x}\right] f''(s)\,e^{\rho s}\,ds.\end{aligned}$$

The notation is abbreviated, because the integrand on the right is the same throughout. It is understood that each path of integration is rectilinear.

Turn now to the other of the two integrals in A. In the first place,

$$(16)\qquad \int_0^\pi f''(s)\,\delta_1(\overline{\rho\pi - s})\,ds = \left[\int_0^x + \int_x^\pi\right] f''(s)\,\delta_1(\overline{\rho\pi - s})\,ds.$$

The integral from x to π may be allowed to stand. The integral from 0 to x is similar to the one just treated. By the same line of reasoning it is found that

$$\begin{aligned}(17)\qquad &\int_0^x f''(s)\,\delta_1(\overline{\rho\pi - s})\,ds\\ &= \left[\delta(\rho\pi)\int_0^y + e^{-\rho\pi}\int_y^x + \omega e^{-\omega\rho\pi}\int_y^{\omega x} + \omega^2 e^{-\omega^2\rho\pi}\int_y^{\omega^2 x}\right] f''(s)\,e^{\rho s}ds.\end{aligned}$$

The analytic character of $f(s)$ is used again, but only in the same region as before.

If the value of A is calculated now, it is seen that the expression

$$\delta(\rho x)\,\delta(\rho\pi)\int_0^y f''(s)\,e^{\rho s}\,ds$$

occurs twice, with opposite signs, and is eliminated. *This is the critical point in the convergence proof.* The remaining terms are harmless by reason of their order of magnitude.

Let B stand for the right-hand member of (15), with the omission of the term containing $\delta(\rho x)$, C for the integral from x to π in (16), and D for the right-hand member of (17), without the term containing $\delta(\rho\pi)$. Then

$$\frac{A}{\rho^2\,\delta(\rho\pi)}=\frac{1}{\rho^2}\left[B-C\frac{\delta(\rho x)}{\delta(\rho\pi)}-D\frac{\delta(\rho x)}{\delta(\rho\pi)}\right].$$

The dependence of B and D on ρ and x is somewhat complicated because of the variability of y, but they are at any rate continuous functions of their arguments, so that there is no difficulty as to the existence of the integrals with regard to ρ.

In the first of the three integrals in B, as s goes along a straight path from y to x, ρs goes along a straight path from ρy to ρx, that is, from $-|\rho x|$ to ρx, for each value of ρ. The real part of ρx is certainly not less than $-|\rho x|$, and so $|e^{\rho s}|$ reaches its greatest value, as far as this integral is concerned, at the upper limit of integration. That is, $|e^{\rho s}|\leqq|e^{\rho x}|$, all along the path. Since the length of the path can not exceed $2x_0$, it follows that if M is the maximum of $|f''(s)|$ for $|s|\leqq x_0$,

$$\left|e^{-\rho x}\int_y^x f''(s)\,e^{\rho s}\,ds\right|\leqq 2Mx_0.$$

Similar reasoning applies to each of the other integrals in B; the definition of y was expressly chosen so as to make the factor $e^{\rho s}$ reach its maximum absolute value at the upper limit of integration in each case. By combination of the inequalities for the three integrals,

$$|B|\leqq 6Mx_0,$$

for all values of x in $(0, x_0)$ and for all values of ρ. Since B is uniformly bounded, and since the length of the path of the ρ-integration is proportional only to the first power of ρ, the integral of B/ρ^2 converges uniformly to zero.

In connection with C, it is enough to consider the ρ-integral along γ_n'', since $C\delta(\rho x)/[\rho^2\,\delta(\rho\pi)]$ takes on conjugate imaginary values for conjugate imaginary values of ρ. The absolute value of $\delta_1(\rho\pi-s)$ can not exceed $3|e^{\omega\rho(s-\pi)}|$, and as s is real here and always $\geqq x$, the expression just written down can not exceed $3|e^{\omega\rho(x-\pi)}|$. Then

$$|C|\leqq 3M\pi\,|e^{\omega\rho(x-\pi)}|, \tag{18}$$

the length of the path being $\leqq \pi$. On the other hand,

$$|\delta(\rho x)| \leqq 3|e^{-\omega\rho x}|,$$

whence

$$|C\delta(\rho x)| \leqq 9M\pi|e^{-\omega\rho\pi}|.$$

But

$$|\delta(\rho\pi)| > k|e^{-\omega\rho\pi}|,$$

where k has the same meaning as in the preceding section. So $C\delta(\rho x)/\delta(\rho\pi)$ is uniformly bounded, and when divided by ρ^2 becomes so small that its ρ-integral approaches zero.

In D, finally, the choice of y is effective in the same way as in B. The three integrals are less than or equal to

$$2Mx_0|e^{\rho x}|, \qquad 2Mx_0|e^{\omega\rho x}|, \qquad 2Mx_0|e^{\omega^2\rho x}|,$$

and the three terms containing them are less than or equal to

$$2Mx_0|e^{\rho(x-\pi)}|, \qquad 2Mx_0|e^{\omega\rho(x-\pi)}|, \qquad 2Mx_0|e^{\omega^2\rho(x-\pi)}|,$$

respectively, in absolute value. Again we may suppose that ρ is on γ_n''; then the second of the last three absolute values is at least as great as either of the others, and

$$|D| \leqq 6Mx_0|e^{\omega\rho(x-\pi)}|.$$

This inequality may be used in exactly the same way as the inequality (18) for $|C|$. It leads to the conclusion that the integral of $D\delta(\rho x)/[\rho^2\delta(\rho\pi)]$ uniformly approaches zero.

Thus the proof of Theorem III is completed.*

* In connection with the present account, attention should be called to the following facts, not brought out by Mr. Hopkins's own treatment: If $f(x)$ is analytic throughout the circle $|x| \leqq \pi$ in the complex plane ($x_0 = \pi$), the convergence of the integral of $A/[\rho^2\delta(\rho\pi)]$ to zero is uniform throughout the entire interval $(0, \pi)$, and nothing stands in the way of the uniform convergence of the series all the way out to the point π, except the term $3f(\pi)\delta(\rho x)/[\rho\delta(\rho\pi)]$. If $f(\pi) = 0$, this term vanishes, and the series is seen to be uniformly convergent for $0 \leqq x \leqq \pi$. It has been observed already that the sum of the series is zero in any case for $x = \pi$, so that this is the only case in which uniform convergence out to the end of the interval is possible.